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A family of super Schrödinger invariant Chern-Simons matter systems

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Abstract: We investigate non-relativistic limits of the $\mathcal{N}$=3 Chern-Simons matter system in 1+2 dimensions. The relativistic theory can generate several inequivalent super Schödinger invariant theories, depending on the degrees of freedom we choose to retain in the non-relativistic limit. The maximally supersymmetric Schrödinger invariant theory is obtained by keeping all particle degrees of freedom. The other descendants, where particles and anti-particles coexist, are also Schrödinger invariant but preserve less supersymmetries. Thus, we have a family of super Schrödinger invariant field theories produced from the parent relativistic theory.

Keywords: Chern-Simons Theories, Supersymmetric gauge theory, Field Theories in Lower Dimensions, Global Symmetries.

Contents

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1. Introduction

One of the hottest issues in AdS/CFT [1–3] is its application to condensed matter physics (CMP), dubbed as AdS/CMP correspondence. Gravitational duals for condensed matter systems such as superconductors [4, 5], quantum hall effects [6, 7] and entanglement entropy [8] are now proliferating in the recent studies of AdS/CMP correspondence (For related progress and a review see [9–12] and [13] respectively).

While these attempts capture some essential aspects of condensed matter systems in a non-trivial way, a noticeable point is that the systems dual to the gravity solutions are typically relativistic. As a consequence, it is not always realistic to compare them with experimental results since condensed matter systems realized in laboratories are typically non-relativistic. Thus, the next step in the AdS/CFT context is to learn how to realize a non-relativistic (NR) limit.

In this context, it would be important to gain much more insights into NR CFTs [14–20]. As a break-through in this direction, gravity duals of NR CFTs have been proposed and further investigated in the literature [21–28, 30–32, 29, 33]. Among NR CFTs, theories with dynamical exponent $z = 2$ are special, and they enjoy the Schrödinger symmetry [14, 15], which is a NR analog of the relativistic conformal symmetry. The name originates from the fact that the symmetry was initially found as the maximal symmetry of a free Schrödinger equation. From the purely field theory viewpoint, an example of Schrödinger invariant field theories was proposed by Jackiw-Pi [16] based on the NR Chern-Simons matter (CSM) system in 1+2 dimensions.

The Schrödinger symmetry can accommodate supersymmetries, in which case it is enhanced to a super Schrödinger symmetry [34–38] (For super Schrödinger invariant gravity duals see [30]). Super Schrödinger invariant field theories would be important like superconformal field theories in our attempts to embed them into string theory. Recently the index for NR SCFTs has also been formulated [39] and it plays a significant role in classifying the theories. However, as far as we know, the supersymmetric Jackiw-Pi model [40], which is obtained by taking a NR limit of the $\mathcal{N}$=2 CSM system [41], is the only example of super Schrödinger invariant field theories with explicit action. It would be, therefore, important to look for other examples in order to deduce some universal features of Schrödinger symmetry and NR supersymmetry.

In this paper we consider NR limits of the $\mathcal{N}$=3 relativistic CSM system [42]. By taking the standard NR limit with only particle degrees of freedom, a new $\mathcal{N}$=3 super Schrödinger invariant CSM system is presented. By using the Noether method, the generators of the super Schrödinger algebra are constructed. As other examples of NR limits, one may consider the mixed cases where particles and anti-particles coexist. The bosonic Schrödinger symmetry still appears but the number of the preserved supersymmetries is reduced. This statement seems general. The bosonic Schrödinger symmetry is independent of the existence of anti-particles but the supersymmetries are affected by them. We emphasize that the way we take the non-relativistic limit affects the symmetry of the resultant non-relativistic theory, which is quite a novel feature.

Before closing the introduction, we should comment on the advantage to begin with the $\mathcal{N}$=3 CSM system. The $\mathcal{N}$=3 supersymmetries has long been the maximum supersymmetries in the conventional Chern-Simons matter system. This is a peculiarity to the (1+2)-dimensional space-time, related to an ambiguity of spin. If we try to realize $\mathcal{N}$=3 supersymmetries in 1+3 dimensions, then the supersymmetries are inevitably enhanced to $\mathcal{N}$=4. Recent developments have shown that quiver-like gauge theories admit $\mathcal{N} = 4$ or greater with a judicious choice of the gauge group and representations [43–45].[1] Thus, as far as the conventional Chern-Simons matter systems are concerned, it would be possible in principle to realize all of the super Schrödinger invariant relatives starting from the $\mathcal{N} = 3$ theory, though we will not try to make a complete list here.

[1] In particular, one of the theories contains $\mathrm{Sp}(2M) \times O(1) \simeq \mathrm{Sp}(2M)$ single gauge field [44]. We would like to thank S. Lee for pointing out an erronous statement in the earlier version.

This paper is organized as follows. Section 2 is a short review of the relativistic $\mathcal{N}$=3 CSM system. In section 3 we study a NR limit of the $\mathcal{N}$=3 CSM system keeping only the particles and derive a new $\mathcal{N}$=3 super Schrödinger invariant CSM system. In section 4 we discuss other NR limits of the $\mathcal{N}$=3 CSM system by mixing particles and anti-particles. We still find the bosonic Schrödinger symmetry but the number of the preserved supersymmetries is decreased. In section 5 we discuss a consistency of NR limits in detail. Section 6 is devoted to a summary and discussions. In appendix A the dimensional counting is explained. Appendix B describes the details of the spinor rotation.

2. $\mathcal{N} = 3$ relativistic CSM system

Our starting point is the $\mathcal{N}$=3 relativistic CSM system in 1+2 dimensions [42]. We shall give a short review below. Our convention used in this paper is also fixed here.

2.1 The action of the $\mathcal{N} = 3$ relativistic CSM system

The Lagrangian of the $\mathcal{N}$=3 relativistic CSM system [42] is composed of the CS term $\mathcal{L}_{\rm CS}$ and the matter part $\mathcal{L}_{\rm M}$ as follows:[2]

$$
\begin{aligned}
\mathcal{L}_{\rm rel} &= \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm M}\,, \\
\mathcal{L}_{\rm CS} &= \frac{\kappa}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} = \kappa A_0 F_{12} + \frac{\kappa}{2c}\epsilon^{ij}\partial_t A_i A_j \qquad (i,j=1,2)\,, \\
\mathcal{L}_{\rm M} &= -|D_\mu\phi_a|^2 - i\bar{\psi}\gamma^\mu D_\mu\psi - i\bar{\chi}\gamma^\mu D_\mu\chi \\
&\quad -\left(\frac{e^2}{\kappa c^2}\right)^2 |\phi_a|^2\left[|\phi_b|^4 + 2v^2\left(|\phi_1|^2 - |\phi_2|^2\right) + v^4\right] - \frac{e^2}{\kappa c^2}v^2\left(i\bar{\psi}\psi - i\bar{\chi}\chi\right) \\
&\quad +\frac{3e^2}{\kappa c^2}\left(|\phi_1|^2 + |\phi_2|^2\right)\left(i\bar{\psi}\psi + i\bar{\chi}\chi\right) - \frac{4ie^2}{\kappa c^2}\left(\phi_1\bar{\psi} - \phi_2\bar{\chi}\right)\left(\phi_1^*\psi - \phi_2^*\chi\right) \\
&\quad -\frac{ie^2}{\kappa c^2}\left(\phi_1\bar{\psi} - \phi_2\bar{\chi}\right)\left(-i\sigma_1\right)\left(\phi_1\psi^* - \phi_2\chi^*\right) \\
&\quad -\frac{ie^2}{\kappa c^2}\left(\phi_1^*\bar{\psi}^* - \phi_2^*\bar{\chi}^*\right)\left(-i\sigma_1\right)\left(\phi_1^*\psi - \phi_2^*\chi\right)\,.
\end{aligned} \tag{2.1}
$$

The matter action contains two complex scalar fields ϕ_a $(a=1,2)$ and two 2-component complex fermions ψ and χ. From the quadratic parts, we identify the mass parameter m as

$$
m^2c^2 \equiv \left(\frac{e^2}{\kappa c^2}\right)^2 v^4\,. \tag{2.2}
$$

The system is superconformal (at least at classical level) when $v^2 = 0$.

In our convention, the sign of the space-time metric is $(-,+,+)$, and we take the Dirac representation for the gamma matrices,

$$
\gamma^0 = -i\sigma_3\,, \quad \gamma^1 = \sigma_1\,, \quad \gamma^2 = \sigma_2\,. \tag{2.3}
$$

[2] We have recovered the speed of light c explicitly. For dimensional analysis see appendix A.

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The Dirac conjugate and covariant derivative are defined as

$$\bar{\psi} = \psi^\dagger \gamma^0\,, \qquad \bar{\psi}^* = \psi^T \gamma^0\,, \qquad D_i = \partial_i + \frac{ie}{c} A_i\,, \qquad D_0 = \frac{1}{c}\partial_t + \frac{ie}{c} A_0\,. \tag{2.4}$$

We note that [42] used the Majorana representation, but we have switched to the Dirac basis that is convenient in the NR limit. Because of this change of spinor basis,[3] the additional factor $-i\sigma_1$ has appeared in the last two lines of (2.1). The σ_1 combines with $\gamma^0 = -i\sigma_3$ in the Dirac conjugate, reducing to the standard epsilon tensor to make a Lorentz scalar out of two (complex conjugated) fermions.

2.2 Supersymmetries

The relativistic $\mathcal{N}$=3 supersymmetries are given by

$$\begin{aligned}
\delta A_\mu &= \frac{e}{\kappa c}\bar{\alpha}_1\gamma_\mu(\psi\phi_2^* - i\sigma_1\chi^*\phi_1) + \frac{e}{\kappa c}\bar{\alpha}_1^* i\sigma_1\gamma_\mu\,(-i\sigma_1\psi^*\phi_2 + \chi\phi_1^*) \\
&\quad + \frac{e}{\kappa c}\bar{\alpha}_2\gamma_\mu(-\psi\phi_1^* - i\sigma_1\psi^*\phi_1 - \chi\phi_2^* - i\sigma_1\chi^*\phi_2)\,, && (2.5)\\
\delta\phi_1 &= i(-\bar{\alpha}_1^* i\sigma_1\chi + \bar{\alpha}_2\psi)\,, && (2.6)\\
\delta\phi_2 &= i(-\bar{\alpha}_1\psi + \bar{\alpha}_2\chi)\,, && (2.7)\\
\delta\psi &= -\gamma^\mu\alpha_1 D_\mu\phi_2 + \gamma^\mu\alpha_2 D_\mu\phi_1 + \frac{e^2}{\kappa c^2}(\alpha_1\phi_2 - \alpha_2\phi_1)(v^2 + |\phi_1|^2 - |\phi_2|^2) \\
&\quad + \frac{2e^2}{\kappa c^2}(-i\sigma_1\alpha_1^*(\phi_1)^2\phi_2^* + \alpha_2\phi_1|\phi_2|^2)\,, && (2.8)\\
\delta\chi &= -i\gamma^\mu\sigma_1\alpha_1^* D_\mu\phi_1 + \gamma^\mu\alpha_2 D_\mu\phi_2 + \frac{e^2}{\kappa c^2}(-i\sigma_1\alpha_1^*\phi_1 + \alpha_2\phi_2)(v^2 + |\phi_1|^2 - |\phi_2|^2) \\
&\quad + \frac{2e^2}{\kappa c^2}(i\sigma_1\alpha_1^*\phi_1^*(\phi_2)^2 + \alpha_2|\phi_1|^2\phi_2)\,. && (2.9)
\end{aligned}$$

The above supersymmetry transformation has been adjusted to our notation from [42].

We write the 2-component complex complex spinors as

$$\alpha_a = \begin{pmatrix}\alpha_a^{(1)}\\ \alpha_a^{(2)}\end{pmatrix} \qquad (a = 1, 2)\,,$$

and we impose the Majorana condition on α_2, which relates the first and second components as

$$\alpha_2^{(2)} = -i(\alpha_2^{(1)})^*\,. \tag{2.10}$$

The number of independent components for α_2 is 2 in real ($\mathcal{N}$=1 in 1+2 dimensions). On the other hand, the number of independent components of α_1 is 4 in real ($\mathcal{N}$=2 in 1+2 dimensions) and corresponds to the $\mathcal{N}$=2 supersymmetries of [40]. In total, the CSM system with the Lagrangian (2.1) has $\mathcal{N}$=3 supersymmetries.

[3] For the details of the spinor rotation see appendix B.

3. NR limit of $\mathcal{N}$=3 CSM — All particle case

Let us discuss a non-relativistic limit of the $\mathcal{N}$=3 CSM system. The CS term is not modified via the NR limit, so we will not touch it for a while, and we concentrate on the matter part only.

With the mass parameter m defined in (2.2), the matter fields are expanded as

$$\begin{aligned}
\phi_a &= \frac{1}{\sqrt{2m}}\left[\mathrm{e}^{-imc^2t}\Phi_a + \mathrm{e}^{imc^2t}\hat{\Phi}_a^*\right] \qquad (a=1,2)\,,\\
\psi &= \sqrt{c}\left[\mathrm{e}^{-imc^2t}\Psi + \mathrm{e}^{imc^2t}C\hat{\Psi}^*\right]\,,\\
\chi &= \sqrt{c}\left[\mathrm{e}^{-imc^2t}\Upsilon + \mathrm{e}^{imc^2t}C\hat{\Upsilon}^*\right]\,.
\end{aligned} \tag{3.1}$$

The symbol "hat" implies anti-particle and $C = i\sigma_2$ is a charge conjugation matrix.

There are several ways to take a NR limit in accordance with the content of the degrees of freedom held in the limit. Here we shall take a natural choice: all of the particles are held on and all of the anti-particles are discarded. That is,

$$\hat{\Phi}_a = \hat{\Psi} = \hat{\Upsilon} = 0\,.$$

The truncation should be consistent and we will show that this limit is indeed a consistent one in section 5.

The NR limit is obtained by substituting the decomposed fields (3.1) into (2.1) and then taking $c \to \infty$ limit. The oscillating terms can be ignored and the Lagrangian is expanded in terms of $1/c$. The resulting first order Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{\mathrm{NR}} = {}& i\Phi_1^* D_t\Phi_1 - \frac{1}{2m}(D_i\Phi_1)^*(D_i\Phi_1) + i\Phi_2^* D_t\Phi_2 - \frac{1}{2m}(D_i\Phi_2)^*(D_i\Phi_2)\\
&+ i\Psi_1^* D_t\Psi_1 - \frac{1}{2m}(D_i\Psi_1)^*(D_i\Psi_1) + i\Upsilon_2^* D_t\Upsilon_2 - \frac{1}{2m}(D_i\Upsilon_2)^*(D_i\Upsilon_2)\\
&- \frac{e}{2mc}F_{12}(|\Psi_1|^2 - |\Upsilon_2|^2) - \lambda\left(|\Phi_1|^4 - |\Phi_2|^4\right)\\
&+ 3\lambda\left(|\Phi_1|^2 + |\Phi_2|^2\right)\left(|\Psi_1|^2 - |\Upsilon_2|^2\right) - 4\lambda\left(|\Phi_1|^2|\Psi_1|^2 - |\Phi_2|^2|\Upsilon_2|^2\right)\\
&- 2i\lambda\left(\Phi_1\Phi_2\Psi_1^*\Upsilon_2^* + \Phi_1^*\Phi_2^*\Psi_1\Upsilon_2\right) + \mathcal{O}(1/c^2)\,.
\end{aligned} \tag{3.2}$$

Here the following quantities have been introduced

$$D_t \equiv cD_0 = \partial_t + ieA_0\,, \qquad \lambda \equiv \frac{e^2}{2m\kappa c}\,.$$

The absolute value of the fermions depends on the ordering. We define it as

$$|\Psi_1|^2 \equiv \Psi_1^*\Psi_1\,, \qquad |\Upsilon_2|^2 \equiv \Upsilon_2^*\Upsilon_2\,. \tag{3.3}$$

In the derivation of (3.2) we used the fermion equations of motion

$$\Psi_2 = -\frac{1}{2mc}D_+\Psi_1 + \mathcal{O}(1/c^2)\,, \qquad \Upsilon_1 = \frac{1}{2mc}D_-\Upsilon_2 + \mathcal{O}(1/c^2)\,, \tag{3.4}$$

and removed Ψ_2 and Υ_1. Here we have recombined the spatial covariant derivatives as follows:

$$D_{\pm} \equiv D_1 \pm iD_2 \,. \tag{3.5}$$

Hereafter we ignore the second order and higher order corrections in terms of $1/c$.

For later purposes, we note that the Lagrangian contains an $\mathcal{N}=2$ NR CSM theory [40] as a subsystem by setting

$$\Phi_1 = \Upsilon_2 = 0\,,$$

up to the difference in conventions.

3.1 Schrödinger symmetry

It is turn to check symmetries of the NR Lagrangian, $\mathcal{L}_{\rm CS} + \mathcal{L}_{\rm NR}$, with (3.2). We first show that the NR Lagrangian possesses a Schrödinger symmetry.

The generators of Schrödinger symmetry. The transformation laws and corresponding charges are summarized below:

1. time translation - $\delta t = -a$
   The transformation law is

$$\begin{aligned}
&\delta\Phi_1 = aD_t\Phi_1\,, \quad \delta\Phi_2 = aD_t\Phi_2\,, \quad \delta\Psi_1 = aD_t\Psi_1\,, \quad \delta\Upsilon_2 = aD_t\Upsilon_2\,,\\
&\delta A_0 = 0\,, \qquad\quad \delta A_i = acF_{0i}\,,
\end{aligned}$$

   and the generator is the Hamiltonian:

$$\begin{aligned}
H = \int d^2x \Big[&\frac{1}{2m}(D_i\Phi_1)^*(D_i\Phi_1) + \frac{1}{2m}(D_i\Phi_2)^*(D_i\Phi_2)\\
&+\frac{1}{2m}(D_i\Psi_1)^*(D_i\Psi_1) + \frac{1}{2m}(D_i\Upsilon_2)^*(D_i\Upsilon_2)\\
&+\frac{e}{2mc}F_{12}(|\Psi_1|^2 - |\Upsilon_2|^2) + \lambda\left(|\Phi_1|^4 - |\Phi_2|^4\right)\\
&-3\lambda\left(|\Phi_1|^2 + |\Phi_2|^2\right)\left(|\Psi_1|^2 - |\Upsilon_2|^2\right) + 4\lambda\left(|\Phi_1|^2|\Psi_1|^2 - |\Phi_2|^2|\Upsilon_2|^2\right)\\
&+2i\lambda\left(\Phi_1\Phi_2\Psi_1^*\Upsilon_2^* + \Phi_1^*\Phi_2^*\Psi_1\Upsilon_2\right)\Big]\,.
\end{aligned} \tag{3.6}$$

2. spatial translation - $\delta x^i = a^i$ $(i=1,2)$
   The transformation law is

$$\begin{aligned}
&\delta\Phi_1 = -a^iD_i\Phi_1\,, \quad \delta\Phi_2 = -a^iD_i\Phi_2\,, \quad \delta\Psi_1 = -a^iD_i\Psi_1\,, \quad \delta\Upsilon_2 = -a^iD_i\Upsilon_2\,,\\
&\delta A_0 = a^iF_{0i}\,, \quad \delta A_i = \epsilon_{ij}a^jF_{12} \qquad (\epsilon_{12} = -\epsilon_{21} = 1)\,,
\end{aligned}$$

   and the generator is the momentum:

$$\begin{aligned}
P_i &= \int d^2x\, p_i\,,\\
p_i &\equiv -\frac{i}{2}\big[\Phi_1^*D_i\Phi_1 - (D_i\Phi_1)^*\Phi_1 + \Phi_2^*D_i\Phi_2 - (D_i\Phi_2)^*\Phi_2\\
&\qquad\quad +\Psi_1^*D_i\Psi_1 - (D_i\Psi_1)^*\Psi_1 + \Upsilon_2^*D_i\Upsilon_2 - (D_i\Upsilon_2)^*\Upsilon_2\big]\,.
\end{aligned} \tag{3.7}$$

3. spatial rotation - $\delta x^i = \theta \epsilon^{ij} x^j$
   The transformation law is

$$\begin{aligned} \delta\Phi_1 &= -\theta\epsilon_{ij}x^i D_j \Phi_1 \,, & \delta\Phi_2 &= -\theta\epsilon_{ij}x^i D_j \Phi_2 \,, \\ \delta\Psi_1 &= -\theta\epsilon_{ij}x^i D_j \Psi_1 - \frac{i}{2}\theta\Psi_1 \,, & \delta\Upsilon_2 &= -\theta\epsilon_{ij}x^i D_j \Upsilon_2 + \frac{i}{2}\theta\Upsilon_2 \,, \\ \delta A_0 &= \theta\epsilon_{ij}x^i F_{0j} \,, & \delta A_i &= \theta x^i F_{12} \,, \end{aligned}$$

   and the generator is the angular momentum:

$$J = \int d^2x \left[\epsilon^{ij} x_i p_j + \frac{1}{2}(|\Psi_1|^2 - |\Upsilon_2|^2) \right] . \tag{3.8}$$

   Note that for the fermionic fields the spin operators are contained. The relative sign of the spin part are fixed from the last term in the Lagrangian (3.2).

4. Galilean boost - $\delta x^i = v^i t$
   The transformation law is

$$\begin{aligned} \delta\Phi_1 &= (imv^i x^i - tv^i D_i)\Phi_1 \,, & \delta\Phi_2 &= (imv^i x^i - tv^i D_i)\Phi_2 \,, \\ \delta\Psi_1 &= (imv^i x^i - tv^i D_i)\Psi_1 \,, & \delta\Upsilon_2 &= (imv^i x^i - tv^i D_i)\Upsilon_2 \,, \\ \delta A_0 &= tv^i F_{0i} \,, & \delta A_i &= t\epsilon_{ij} v^j F_{12} \,, \end{aligned}$$

   and the corresponding generator is

$$G_i = tP_i - m \int d^2x \, x_i \rho \,. \tag{3.9}$$

5. dilatation - $\delta t = 2at \,, \;\; \delta x^i = ax^i$
   The transformation law is

$$\begin{aligned} \delta\Phi_1 &= -a\left[1 + x^i D_i + 2tD_t\right]\Phi_1 \,, & \delta\Phi_2 &= -a\left[1 + x^i D_i + 2tD_t\right]\Phi_2 \,, \\ \delta\Psi_1 &= -a\left[1 + x^i D_i + 2tD_t\right]\Psi_1 \,, & \delta\Upsilon_2 &= -a\left[1 + x^i D_i + 2tD_t\right]\Upsilon_2 \,, \\ \delta A_0 &= ax^i F_{0i} \,, & \delta A_i &= a(\epsilon_{ij}x^j F_{12} - 2ctF_{0i}) \,. \end{aligned}$$

   The corresponding generator is

$$D = -2tH + \int d^2x \, x^i p_i \,. \tag{3.10}$$

	Φ_1	Φ_2	Ψ_1	Υ_2
U(1)$_1$	0	1	1	0
U(1)$_2$	1	0	0	1
U(1)$_3$	1	0	1	0
U(1)$_4$	0	1	0	1

Table 1: The four U(1) symmetries

6. special conformal transformation - $\delta t = at^2\,,\ \delta x^i = atx^i$
   The transformation law is

$$\begin{aligned}
\delta\Phi_1 &= -\left(at - \frac{i}{2}ma(x^i)^2 + atx^iD_i + at^2D_t\right)\Phi_1\,,\\
\delta\Phi_2 &= -\left(at - \frac{i}{2}ma(x^i)^2 + atx^iD_i + at^2D_t\right)\Phi_2\,,\\
\delta\Psi_1 &= -\left(at - \frac{i}{2}ma(x^i)^2 + atx^iD_i + at^2D_t\right)\Psi_1\,,\\
\delta\Upsilon_2 &= -\left(at - \frac{i}{2}ma(x^i)^2 + atx^iD_i + at^2D_t\right)\Upsilon_2\,,\\
\delta A_0 &= atx^iF_{0i}\,,\\
\delta A_i &= at\epsilon_{ij}x^jF_{12} - at^2cF_{0i}\,,
\end{aligned} \tag{3.11}$$

   and the corresponding generator is

$$K = t^2H + tD - \frac{1}{2}m\int d^2x\,(x^i)^2\rho\,, \tag{3.12}$$

   where ρ is a particle density and defined as

$$\rho \equiv |\Phi_1|^2 + |\Phi_2|^2 + |\Psi_1|^2 + |\Upsilon_2|^2\,. \tag{3.13}$$

   Note that it carries no space-time index in comparison with the relativistic conformal case.

U(1) symmetries. In addition to the above generators, the mass operator

$$M = m\int d^2x\,\rho \tag{3.14}$$

is also a conserved quantity as a part of the Galilean algebra (or more precisely Bargmann algebra). Here ρ is the number density and its integral is the total number. Hence M just gives the total mass.

The conservation of M is related to U(1) symmetries. In this case the following four U(1)s may be found (see table 1).

Thus, we have the four conserved quantities:

$$\begin{aligned}
I_1 &= \int d^2x\,\left(|\Phi_2|^2 + |\Psi_1|^2\right)\,, \quad I_2 = \int d^2x\,\left(|\Phi_1|^2 + |\Upsilon_2|^2\right)\,,\\
I_3 &= \int d^2x\,\left(|\Phi_1|^2 + |\Psi_1|^2\right)\,, \quad I_4 = \int d^2x\,\left(|\Phi_2|^2 + |\Upsilon_2|^2\right)\,.
\end{aligned} \tag{3.15}$$

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The conservation of M follows from $I_1 + I_2$.

Note that one of U(1)'s can be absorbed by using the simultaneous phase transformation (i.e. $I_1 + I_2 = I_3 + I_4$), so there remains $\mathrm{U}(1)^3$. By taking a linear combination of (3.15), we can fix the three U(1)'s. One of them should be the mass parameter M. We choose the remaining two as follows:

$$N_{\rm B} = \int d^2x \, \left(|\Phi_2|^2 - |\Phi_1|^2\right) , \tag{3.16}$$

$$N_{\rm F} = \int d^2x \, \left(|\Psi_1|^2 - |\Upsilon_2|^2\right) . \tag{3.17}$$

We have seen that $N_{\rm F}$ generates the spin from the expression (3.8). We will see that supercharges are eigenstates of $N_{\rm B}$ and $N_{\rm F}$.

The Poisson brackets. In order to study the algebra, we compute the Poisson brackets of the above generators. For the matter fields, by using the classical Poisson brackets

$$\begin{aligned}
\{\Phi_1(x), \Phi_1^*(x')\}_{\rm PB} &= -\{\Phi_1^*(x), \Phi_1(x')\}_{\rm PB} = -i\delta^{(2)}(x - x') , \\
\{\Phi_2(x), \Phi_2^*(x')\}_{\rm PB} &= -\{\Phi_2^*(x), \Phi_2(x')\}_{\rm PB} = -i\delta^{(2)}(x - x') , \\
\{\Psi_1(x), \Psi_1^*(x')\}_{\rm PB} &= \{\Psi_1^*(x), \Psi_1(x')\}_{\rm PB} = -i\delta^{(2)}(x - x') , \\
\{\Upsilon_2(x), \Upsilon_2^*(x')\}_{\rm PB} &= \{\Upsilon_2^*(x), \Upsilon_2(x')\}_{\rm PB} = -i\delta^{(2)}(x - x') ,
\end{aligned}$$

one can compute the Poisson brackets of the bosonic generators

$$H, \; P_i, \; J, \; G_i, \; D, \; K, \; M, \; N_{\rm B}, \; N_{\rm F} . \tag{3.18}$$

The number of the generators is 11 $(= 1 + 2 + 1 + 2 + 1 + 1 + 1 + 1 + 1)$. Henceforth we represent the Poisson bracket for bosonic (fermionic) generators by [,] ({ , }).

The treatment of the gauge field is more involved. Note that A_i appears in the generators but A_0 does not. For A_i, by solving the equation of motion for A_0 (the Gauss law constraint),

$$F_{12} = \frac{e}{\kappa}\rho ,$$

we can obtain the explicit expression,

$$A_i(t, x^i) = -\frac{e}{\kappa}\epsilon_{ij}\partial_j \int d^2y \, G(x - y)\rho(t, y^i) , \tag{3.19}$$

$$G(x - y) = \frac{1}{2\pi} \ln|x - y| , \qquad \partial_i^2 G(x - y) = \delta^{(2)}(x - y) . \tag{3.20}$$

Using (3.19) we can compute the Poisson bracket including A_i.

The Schrödinger algebra. The resulting algebra is

$$\begin{aligned}
&[P_i, P_j] = [P_i, H] = [J, H] = [G_i, G_j] = 0 , \\
&[J, P_i] = \epsilon_{ij}P_j , \qquad [J, G_i] = \epsilon_{ij}G_j , \qquad [P_i, G_j] = \delta_{ij}M , \qquad [G_i, H] = -P_i , \\
&[D, H] = 2H , \qquad [D, K] = -2K , \qquad [K, H] = -D ,
\end{aligned}$$

$$
\begin{aligned}
&[D,P_i]=P_i\,, && [D,J]=0\,, && [D,G_i]=-G_i\,,\\
&[K,P_i]=G_i\,, && [K,J]=[K,G_i]=0\,, && [M,*]=[N_{\rm B},*]=[N_{\rm F},*]=0\,,
\end{aligned}
\qquad (3.21)
$$

where the symbol $*$ in the Poisson brackets imply any bosonic generators. This is the Schrödinger algebra. The three Poisson brackets in the third line of (3.21) describe the algebra of the one-dimensional conformal group $\mathrm{SO}(2,1)$. The algebra (3.21) contains the Bargmann algebra spanned by $\{H,P_i,J,G_i,M\}$ as a subalgebra. This is a central extension of the Galilean algebra spanned by $\{H,P_i,J,G_i\}$ with the mass generator M .

It is easy to check the conservation of the generators with (3.21) and the following Hamilton equation:

$$
\frac{dA}{dt}=\frac{\partial A}{\partial t}+[A,H] \qquad (A:\ \text{any generator})\,. \qquad (3.22)
$$

Now that G_i, D and K explicitly depend on the time t , they do not commute with the Hamiltonian H but are still conserved.

3.2 Supersymmetries

Let us consider a non-relativistic limit of the original relativistic supersymmetries. In the non-relativistic limit, the relativistic transformation law can be expanded in terms of c . The non-relativistic analog of supersymmetry transformations are determined order by order. For this purpose it is helpful to recall that

$$
v^2=\frac{\kappa c^3 m}{e^2} \quad \text{and} \quad A_0\sim\mathcal{O}(1/c)\,.
$$

The NR supersymmetries. The supersymmetry transformation at the leading order is given by

$$
\begin{aligned}
\delta_1\Phi_1 &= \sqrt{2mc}\,(-i\alpha_1^{(1)}\Upsilon_2+\alpha_2^{(1)*}\Psi_1)\,,\\
\delta_1\Phi_2 &= -\sqrt{2mc}\,(\alpha_1^{(1)*}\Psi_1+\alpha_2^{(2)*}\Upsilon_2)\,,\\
\delta_1\Psi_1 &= \sqrt{2mc}\,(\alpha_1^{(1)}\Phi_2-\alpha_2^{(1)}\Phi_1)\,,\\
\delta_1\Upsilon_2 &= \sqrt{2mc}\,(-i\alpha_1^{(1)*}\Phi_1+\alpha_2^{(2)}\Phi_2)\,,\\
\delta_1 A_0 &= \frac{e}{\sqrt{2mc}\,\kappa}\Big[\alpha_1^{(1)*}\Psi_1\Phi_2^*-i\alpha_1^{(1)*}\Upsilon_2^*\Phi_1-\alpha_1^{(1)}\Psi_1^*\Phi_2-i\alpha_1^{(1)}\Upsilon_2\Phi_1^*\\
&\qquad\qquad -\alpha_2^{(1)*}\Psi_1\Phi_1^*-i\alpha_2^{(2)*}\Psi_1^*\Phi_1-\alpha_2^{(2)*}\Upsilon_2\Phi_2^*-i\alpha_2^{(1)*}\Upsilon_2^*\Phi_2\Big]\,,\\
\delta_1 A_i &= 0\,.
\end{aligned}
$$

Note that the reality of A_0 is preserved under the condition (2.10) . The leading supersymmetry is often called the kinematical supersymmetry.

The second supersymmetry transformation is obtained from the next-to-leading order in the NR limit and given by

$$
\begin{aligned}
\delta_2\Phi_1 &= \frac{1}{\sqrt{2mc}}(\alpha_2^{(2)*}D_+\Psi_1+i\alpha_1^{(2)}D_-\Upsilon_2)\,,\\
\delta_2\Phi_2 &= \frac{1}{\sqrt{2mc}}(-\alpha_1^{(2)*}D_+\Psi_1+\alpha_2^{(1)*}D_-\Upsilon_2)\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta_2\Psi_1 &= -\frac{1}{\sqrt{2mc}}(\alpha_1^{(2)}D_-\Phi_2 - \alpha_2^{(2)}D_-\Phi_1)\,,\\
\delta_2\Upsilon_2 &= \frac{1}{\sqrt{2mc}}(-i\alpha_1^{(2)*}D_+\Phi_1 + \alpha_2^{(1)}D_+\Phi_2)\,,\\
\delta_2 A_0 &= \frac{e}{(2mc)^{3/2}\kappa}\Big[-\alpha_1^{(2)*}(D_+\Psi_1)\Phi_2^* - i\alpha_1^{(2)*}(D_-\Upsilon_2)^*\Phi_1\\
&\qquad\qquad\qquad +\alpha_1^{(2)}(D_+\Psi_1)^*\Phi_2 - i\alpha_1^{(2)}(D_-\Upsilon_2)\Phi_1^*\\
&\qquad\qquad\qquad +\alpha_2^{(2)*}(D_+\Psi_1)\Phi_1^* + i\alpha_2^{(1)*}(D_+\Psi_1)^*\Phi_1\\
&\qquad\qquad\qquad -\alpha_2^{(1)*}(D_-\Upsilon_2)\Phi_2^* - i\alpha_2^{(2)*}(D_-\Upsilon_2)^*\Phi_2\Big]\,,\\
\delta_2 A_+ &= \frac{2ie}{\sqrt{2mc}\,\kappa}\Big[\alpha_1^{(2)}\Psi_1^*\Phi_2 + i\alpha_1^{(2)}\Upsilon_2\Phi_1^* + i\alpha_2^{(1)*}\Psi_1^*\Phi_1 + \alpha_2^{(1)*}\Upsilon_2\Phi_2^*\Big]\,,\\
\delta_2 A_- &= -\frac{2ie}{\sqrt{2mc}\,\kappa}\Big[-\alpha_1^{(2)*}\Psi_1\Phi_2^* + i\alpha_1^{(2)*}\Upsilon_2^*\Phi_1 + \alpha_2^{(2)*}\Psi_1\Phi_1^* + i\alpha_2^{(2)*}\Upsilon_2^*\Phi_2\Big]\,.
\end{aligned}
$$

The next-to-leading supersymmetry is often called the dynamical supersymmetry.

Here we should notice that $\alpha_1^{(1)}$ and $\alpha_1^{(2)}$ are completely separated as in the case of [40] (actually those correspond to $\mathcal{N}$=2 of [40]), but $\alpha_2^{(1)}$ (equivalently $\alpha_2^{(2)}$) appears in both the leading and the next-to-leading supersymmetries. One might naively expect that the NR supersymmetry should be enhanced after taking the NR limit. It is not the case, however. By directly checking the symmetry, we can realize that the leading supersymmetry is preserved while the next-to-leading one (for α_2) is broken due to the presence of the interaction potential. Indeed, the last potential term in (3.2) breaks the symmetry.[4]

Supercharges. By using the Noether method, we can construct supercharges corresponding to the above supersymmetry transformations.

The supercharges for the two leading supersymmetries are

$$Q_1^{(1)} = \sqrt{2m}\int d^2x\ [\Phi_1^*\Upsilon_2 - i\Phi_2\Psi_1^*]\ , \tag{3.23}$$

$$Q_1^{(2)} = \sqrt{2m}\int d^2x\ [\Phi_2^*\Upsilon_2 + i\Phi_1\Psi_1^*]\ , \tag{3.24}$$

(up to rescaling $\sqrt{c}$) and the supersymmetry transformations for the matter fields are generated by, respectively,

$$\alpha_1^{(1)}Q_1^{(1)} + Q_1^{(1)*}\alpha_1^{(1)*}\,, \qquad \alpha_2^{(1)}Q_1^{(2)} + Q_1^{(2)*}\alpha_2^{(1)*}\,.$$

For example, the transformation in terms of the first charge $\delta_1^{(1)}$ for Φ_1 is given by

$$\delta_1^{(1)}\Phi_1 = [\Phi_1, \alpha_1^{(1)}Q_1^{(1)} + Q_1^{(1)*}\alpha_1^{(1)*}]\,.$$

The next-to-leading supercharge is given by

$$Q_2 = \frac{1}{\sqrt{2m}}\int d^2x\ [-\Phi_1^*D_-\Upsilon_2 - i\Phi_2(D_+\Psi_1)^*]\ , \tag{3.25}$$

[4] In the free field theory limit, the next-to-leading supersymmetry is a symmetry, but the algebra does not close as a conventional supersymmetry algebra.

(up to rescaling $1/\sqrt{c}$) and the transformation for the matter fields are generated by

$$\alpha_1^{(2)} Q_2 + Q_2^* \alpha_1^{(2)*} \,.$$

The algebra with supercharges. The Poisson brackets including supercharges only are

$$\begin{aligned}
&\{Q_1^{(1)}, Q_1^{(1)*}\} = \{Q_1^{(2)}, Q_1^{(2)*}\} = -2iM \,, && \{Q_1^{(1)}, Q_1^{(2)}\} = \{Q_1^{(1)}, Q_1^{(2)*}\} = 0 \,, \\
&\{Q_1^{(1)}, Q_2\} = 0 \,, && \{Q_1^{(1)}, Q_2^*\} = P_+ \,, \\
&\{Q_1^{(2)}, Q_2\} = \{Q_1^{(2)}, Q_2^*\} = 0 \,, && \{Q_2, Q_2^*\} = -iH \,.
\end{aligned} \tag{3.26}$$

In the derivation of the last Poisson bracket, we have used the Gauss law constraint

$$F_{12} = \frac{e}{\kappa} \left(|\Phi_1|^2 + |\Phi_2|^2 + |\Psi_1|^2 + |\Upsilon_2|^2 \right) \,. \tag{3.27}$$

The Poisson brackets of the bosonic generators $(H, P_i, J, G_i, M, N_{\rm B}, N_{\rm F})$ and the supercharges are

$$\begin{aligned}
&[P_i, Q_1^{(1)}] = [P_i, Q_1^{(2)}] = [P_i, Q_2] = 0 \,, \quad [H, Q_1^{(1)}] = [H, Q_1^{(2)}] = [H, Q_2] = 0 \,, \\
&[Q_1^{(1)}, J] = \frac{i}{2} Q_1^{(1)} \,, \qquad [Q_1^{(2)}, J] = \frac{i}{2} Q_1^{(2)} \,, \qquad [J, Q_2] = \frac{i}{2} Q_2 \,, \\
&[G_i, Q_1^{(1)}] = [G_i, Q_1^{(2)}] = 0 \,, \qquad [Q_2, G_+] = -iQ_1^{(1)} \,, \qquad [Q_2, G_-] = 0 \,, \\
&[N_{\rm B}, Q_1^{(1)}] = iQ_1^{(1)} \,, \qquad [N_{\rm F}, Q_1^{(1)}] = -iQ_1^{(1)} \,, \qquad [M, Q_1^{(1)}] = 0 \,, \\
&[N_{\rm B}, Q_1^{(2)}] = -iQ_1^{(2)} \,, \qquad [N_{\rm F}, Q_1^{(2)}] = -iQ_1^{(2)} \,, \qquad [M, Q_1^{(2)}] = 0 \,, \\
&[N_{\rm B}, Q_2] = iQ_2 \,, \qquad [N_{\rm F}, Q_2] = -iQ_2 \,, \qquad [M, Q_2] = 0 \,.
\end{aligned} \tag{3.28}$$

The above algebra and the Bargmann algebra give a closed subalgebra called super Bargmann algebra. It is worthwhile mentioning that supercharges are eigenstates of $N_{\rm B}$ and $N_{\rm F}$. Thus $N_{\rm B}$ and $N_{\rm F}$ can be interpreted as R-charges.

Superconformal symmetry. Now there are conformal generators D and K . The following Poisson brackets yield a closed algebra:

$$[D, Q_1^{(1)}] = [K, Q_1^{(1)}] = [D, Q_1^{(2)}] = [K, Q_1^{(2)}] = 0 \,, \quad [D, Q_2] = Q_2 \,, \tag{3.29}$$

but, for the Poisson bracket of K and Q_2 , we have to introduce a new generator S describing superconformal symmetry as follows:

$$[K, Q_2] = -iS \,. \tag{3.30}$$

Here S is explicitly given by

$$S = itQ_2 - \sqrt{\frac{m}{2}} \int d^2x \, x^- \left(\Phi_1^* \Upsilon_2 - i\Phi_2 \Psi_1^* \right) \,, \tag{3.31}$$

where $x^- = x^1 - ix^2$.

While the superconformal transformation for the matter fields is generated by $\beta S + S^*\beta^*$ with the Poisson bracket as $\delta\Phi = [\Phi, \beta S + S^*\beta^*]$, it is not so trivial to fix the gauge field transformation and it has not been done even for the $\mathcal{N}$=2 case [40].

A key observation is to notice that the explicit t-dependence in front of Q_2 in the superconformal charge S gives the additional terms in the superconformal variation in the action, but those are canceled out by the second term contribution in S. This cancellation mechanism works also for the gauge field transformation and this strategy enables us to derive the additional transformation explicitly.

The superconformal transformation law is shown to be

$$\begin{aligned}
\delta\Phi_1 &= -\frac{1}{\sqrt{2m}}\, t\beta D_-\Upsilon_2 + i\sqrt{\frac{m}{2}}\, x^-\beta\Upsilon_2\,, \\
\delta\Phi_2 &= \frac{1}{\sqrt{2m}}\, it\beta^* D_+\Psi_1 + \sqrt{\frac{m}{2}}\, x^+\beta^*\Psi_1\,, \\
\delta\Psi_1 &= -\frac{1}{\sqrt{2m}}\, it\beta D_-\Phi_2 - \sqrt{\frac{m}{2}}\, x^-\beta\Phi_2\,, \\
\delta\Upsilon_2 &= -\frac{1}{\sqrt{2m}}\, t\beta^* D_+\Phi_1 + i\sqrt{\frac{m}{2}}\, x^+\beta^*\Phi_1\,, \\
\delta A_0 &= \frac{e}{(2m)^{3/2}\kappa c}\Big[it\beta^* D_+\Psi_1\Phi_2^* - t\beta^*(D_-\Upsilon_2)^*\Phi_1 \\
&\qquad\qquad +it\beta(D_+\Psi_1)^*\Phi_2 + t\beta D_-\Upsilon_2\Phi_1^* \\
&\qquad\qquad +mx^-\beta(\Phi_2\Psi_1^* + i\Phi_1^*\Upsilon_2) - mx^+\beta^*(\Phi_2^*\Psi_1 - i\Phi_1\Upsilon_2^*)\Big]\,, \\
\delta A_+ &= \frac{2ie}{\sqrt{2m}\,\kappa}\Big[it\beta\Psi_1^*\Phi_2 - t\beta\Upsilon_2\Phi_1^*\Big]\,, \\
\delta A_- &= -\frac{2ie}{\sqrt{2m}\,\kappa}\Big[it\beta^*\Psi_1\Phi_2^* + t\beta^*\Upsilon_2^*\Phi_1\Big]\,.
\end{aligned} \tag{3.32}$$

It is easy to check that the action is indeed invariant under this transformation (3.32). As a side remark, we note that the transformation law above (up on a trivial truncation of fields) gives the missing gauge field transformations in the $\mathcal{N} = 2$ NR CSM system [40] under the superconformal transformation.

The Poisson brackets of S and the bosonic generators are

$$\begin{aligned}
&[S,H] = -iQ_2\,, && [P_+,S] = Q_1^{(1)}\,, && [P_-,S] = 0\,, && [J,S] = \frac{i}{2}S\,, \\
&[S,G_i] = 0\,, && [S,D] = S\,, && [S,K] = 0\,, && \\
&[N_{\mathrm{B}},S] = iS\,, && [N_{\mathrm{F}},S] = -iS\,, && [M,S] = 0\,. &&
\end{aligned} \tag{3.33}$$

The first Poisson bracket indicates the conservation of S as in (3.22).

Similarly, the Poisson brackets with the supercharges are found to be

$$\begin{aligned}
\{S, Q_1^{(1)}\} &= \{S, Q_1^{(2)}\} = \{S, Q_2\} = \{S, Q_1^{(2)*}\} = 0\,, \\
\{S, S^*\} &= iK\,, \\
\{S, Q_1^{(1)*}\} &= -iG_-\,, \\
\{S, Q_2^*\} &= \frac{i}{2}\left[iD - J + N_{\rm B} - \frac{1}{2}N_{\rm F}\right].
\end{aligned} \tag{3.34}$$

Thus, we have shown that a set of the generators

$$H,\ P_i,\ J,\ G_i,\ D,\ K,\ M,\ N_{\rm B},\ N_{\rm F},\ Q_1^{(1)},\ Q_1^{(2)},\ Q_2,\ S$$

spans a super Schrödinger algebra with 8 supercharges (in real components). We would like to emphasize this super Schrödinger algebra has not appeared before in the literature and it is a novel algebra.

3.3 The positivity of the hamiltonian

It is valuable to see the positivity of the Hamiltonian (3.6). From the expression of (3.6), the positivity is quite non-trivial since (3.6) contains a quartic potential with the negative sign. Nevertheless, the Hamiltonian is positive definite as required from the supersymmetry.

With the Gauss law constraint (3.27), we can drastically simplify (3.6) to

$$\begin{aligned}
H = \int d^2x \bigg[&\frac{1}{2m}(D_+\Phi_1)^* D_+\Phi_1 + \frac{1}{2m}(D_-\Phi_2)^* D_-\Phi_2 \\
&+\frac{1}{2m}(D_-\Psi_1)^* D_-\Psi_1 + \frac{1}{2m}(D_+\Upsilon_2)^* D_+\Upsilon_2 \\
&+2\lambda(\Phi_1\Psi_1^* - i\Phi_2^*\Upsilon_2)(\Phi_1^*\Psi_1 + i\Phi_2\Upsilon_2^*)\bigg].
\end{aligned} \tag{3.35}$$

This form of the Hamiltonian is manifestly semi-positive definite. In relation to the $\mathcal{N}=2$ system, we note that by setting $\Phi_1 = \Upsilon_2 = 0$, the last term in (3.35) and the related kinematic terms vanish and the Hamiltonian in [40] can be reproduced.

From the expression (3.35) it is easy to figure out the conditions for the lowest energy solution. Those are given by the familiar ones

$$D_1\Phi_1 = -iD_2\Phi_1\,, \quad D_1\Phi_2 = iD_2\Phi_2\,, \quad D_1\Psi_1 = iD_2\Psi_1\,, \quad D_1\Upsilon_2 = -iD_2\Upsilon_2\,, \tag{3.36}$$

and the additional constraint

$$\Phi_1\Psi_1^* = i\Phi_2^*\Upsilon_2\,. \tag{3.37}$$

As a matter of course, the static soliton solution found in [40] satisfies these conditions when $\Phi_1 = \Upsilon_2 = 0$. However, if we try to turn on Φ_1 and Υ_2 non-trivially, then we cannot fix A_1 and A_2 consistently to Φ_2 and Ψ_1. It would be interesting to look for solutions of different type. For example, spinor vortex solutions are disccused in [46].

4. NR limit of $\mathcal{N}$=3 CSM — mixed cases

In the previous section we have discussed the case that all the particles are kept and all the anti-particles are discarded. However there is no reason why anti-particles are dropped off. Hence we shall consider other NR limits containing anti-particles.

Recall that the matter fields are expanded as in (3.1),

$$\begin{aligned}\phi_a &= \frac{1}{\sqrt{2m}}\left[\mathrm{e}^{-imc^2t}\Phi_a + \mathrm{e}^{imc^2t}\hat{\Phi}_a^*\right] \qquad (a=1,2)\,,\\ \psi &= \sqrt{c}\left[\mathrm{e}^{-imc^2t}\Psi + \mathrm{e}^{imc^2t}C\hat{\Psi}^*\right]\,,\\ \chi &= \sqrt{c}\left[\mathrm{e}^{-imc^2t}\Upsilon + \mathrm{e}^{imc^2t}C\hat{\Upsilon}^*\right]\,.\end{aligned}$$

There are actually several choices to take the NR limits containing anti-particles. We cannot, however, freely choose the matter content kept in the NR limits if we take care of the consistency of the limits with the parent theory. We will discuss the consistency of the NR limits in section 5.

Here we shall pick up the following consistent cases:

1. the APPA case:
$$\Phi_1 = \hat{\Phi}_2 = \hat{\Psi} = \Upsilon = 0$$
2. the PAPA case:
$$\hat{\Phi}_1 = \Phi_2 = \hat{\Psi} = \Upsilon = 0\,.$$

The sequences of alphabets in the items indicate which of particle (P) and anti-particle (A) is picked up in $(\phi_1, \phi_2, \psi, \chi)$, respectively.[5]

We will discuss each of the cases below. We will not touch the CS term again and concentrate on the matter part only.

4.1 A mixed case 1. - the APPA case

Here let us consider the following mixed case:

$$\Phi_1 = \hat{\Phi}_2 = \hat{\Psi} = \Upsilon = 0\,.$$

The matter content leads us to the following NR Lagrangian for the matter fields

$$\begin{aligned}\mathcal{L}_{\rm NR} = & \, i\hat{\Phi}_1^*\hat{D}_t\hat{\Phi}_1 - \frac{1}{2m}(\hat{D}_i\hat{\Phi}_1)^*\hat{D}_i\hat{\Phi}_1 + i\Phi_2^*D_t\Phi_2 - \frac{1}{2m}(D_i\Phi_2)^*D_i\Phi_2\\ & + i\Psi_1^*D_t\Psi_1 - \frac{1}{2m}(D_i\Psi_1)^*D_i\Psi_1 + i\hat{\Upsilon}_2^*\hat{D}_t\hat{\Upsilon}_2 - \frac{1}{2m}(\hat{D}_i\hat{\Upsilon}_2)^*\hat{D}_i\hat{\Upsilon}_2\\ & - \frac{e}{2mc}F_{12}(|\Psi_1|^2 + |\hat{\Upsilon}_2|^2) - \lambda\left(|\hat{\Phi}_1|^4 - |\Phi_2|^4\right)\\ & + 3\lambda\left(|\hat{\Phi}_1|^2 + |\Phi_2|^2\right)\left(|\Psi_1|^2 - |\hat{\Upsilon}_2|^2\right)\\ & - 4\lambda\left(|\hat{\Phi}_1|^2|\Psi_1|^2 - |\Phi_2|^2|\hat{\Upsilon}_2|^2\right) + \mathcal{O}(1/c^2)\,,\end{aligned} \tag{4.1}$$

[5] In principle, we may keep both particle and anti-particle in a single field (B) or neither of them (N). We will not consider here these cases such as BAPN in this paper.

where we have introduced another covariant derivative, which is friendly to anti-particles,

$$\hat{D}_i \equiv \partial_i - \frac{ie}{c}A_i\,, \qquad \hat{D}_t = c\hat{D}_0 = \partial_t - ieA_0\,.$$

In the derivation of (4.1) we used the fermion equations of motion

$$\Psi_2 = -\frac{1}{2mc}D_+\Psi_1\,, \qquad \hat{\Upsilon}_1 = -\frac{1}{2mc}\hat{D}_-\hat{\Upsilon}_2 \tag{4.2}$$

and removed Ψ_2 and $\hat{\Upsilon}_1$. Here we have also recombined $\hat{D}_i$ as

$$\hat{D}_\pm \equiv \hat{D}_1 \pm i\hat{D}_2\,. \tag{4.3}$$

Comparing (4.1) with (3.2), we note that the signs of the charges of anti-particles are flipped and the last terms of (3.2) are missing. Again, one can reproduce the $\mathcal{N}$=2 action in [40] by setting $\hat{\Phi}_1 = \hat{\Upsilon}_2 = 0$.

Schrödinger symmetry. The NR Lagrangian with (4.1) still has the Schrödinger symmetry. The algebra can easily be derived in the same way as in the previous section. For the bosonic generators the difference is just a sign of the charge e for anti-particles, so we will not repeat the computation of the algebra here.

For the spin operators, the relative sign is not fixed since the Lagrangian (4.1) does not contain the term like the last term in (3.2). In the mixed case, however, it is possible to rotate Ψ_1 and $\hat{\Upsilon}_2$ independently and there is an ambiguity for the definition of their spins. Consequently, the undetermined relative sign does not cause any problem.

U(1) symmetries. There are four U(1) symmetries in this case and the number of each of particles and anti-particles is conserved. The corresponding generators are

$$N_{\mathrm{B1}} = \int d^2x\, |\hat{\Phi}_1|^2\,, \quad N_{\mathrm{B2}} = \int d^2x\, |\Phi_2|^2\,,$$
$$N_{\mathrm{F1}} = \int d^2x\, |\Psi_1|^2\,, \quad N_{\mathrm{F2}} = \int d^2x\, |\hat{\Upsilon}_2|^2\,.$$

Note that the mass operator M is proportional to a sum of them and not an independent quantity.

The positivity of the hamiltonian. The original Hamiltonian is given by

$$\begin{aligned} H = \int d^2x \Big[&\frac{1}{2m}(\hat{D}_i\hat{\Phi}_1)^*\hat{D}_i\hat{\Phi}_1 + \frac{1}{2m}(D_i\Phi_2)^*D_i\Phi_2 \\ &+\frac{1}{2m}(D_i\Psi_1)^*D_i\Psi_1 + \frac{1}{2m}(\hat{D}_i\hat{\Upsilon}_2)^*\hat{D}_i\hat{\Upsilon}_2 \\ &+\frac{e}{2mc}F_{12}(|\Psi_1|^2 + |\hat{\Upsilon}_2|^2) + \lambda(|\hat{\Phi}_1|^4 - |\Phi_2|^4) \\ &-3\lambda(|\hat{\Phi}_1|^2 + |\Phi_2|^2)(|\Psi_1|^2 - |\hat{\Upsilon}_2|^2) + 4\lambda(|\hat{\Phi}_1|^2|\Psi_1|^2 - |\Phi_2|^2|\hat{\Upsilon}_2|^2)\Big]\,. \end{aligned} \tag{4.4}$$

By using the Gauss law constraint

$$F_{12} = \frac{e}{\kappa}(-|\hat{\Phi}_1|^2 + |\Phi_2|^2 + |\Psi|^2 - |\hat{\Upsilon}_2|^2)\,, \tag{4.5}$$

this expression can be rewritten as

$$H = \int d^2x \left[\frac{1}{2m}(\hat{D}_+\hat{\Phi}_1)^*\hat{D}_+\hat{\Phi}_1 + \frac{1}{2m}(D_-\Phi_2)^*D_-\Phi_2 \right.$$
$$\left. +\frac{1}{2m}(D_-\Psi_1)^*D_-\Psi_1 + \frac{1}{2m}(\hat{D}_+\hat{\Upsilon}_2)^*\hat{D}_+\hat{\Upsilon}_2\right]. \tag{4.6}$$

Thus, the Hamiltonian is semi-positive definite. The conditions for the lowest energy solution are

$$\hat{D}_+\hat{\Phi}_1 = D_-\Phi_2 = D_-\Psi_1 = \hat{D}_+\hat{\Upsilon}_2 = 0\,. \tag{4.7}$$

Supersymmetries. The supersymmetry transformation at the leading order is given by

$$\begin{aligned}
&\delta_1\hat{\Phi}_1 = i\sqrt{2mc}\,\alpha_1^{(2)*}\hat{\Upsilon}_2\,, \qquad && \delta_1\Phi_2 = -\sqrt{2mc}\,\alpha_1^{(1)*}\Psi_1\,,\\
&\delta_1\Psi_1 = \sqrt{2mc}\,\alpha_1^{(1)}\Phi_2\,, && \delta_1\hat{\Upsilon}_2 = i\sqrt{2mc}\,\alpha_1^{(2)}\hat{\Phi}_1\,,\\
&\delta_1 A_0 = \frac{e}{\sqrt{2mc}\kappa}\left[\alpha_1^{(1)*}\Psi_1\Phi_2^* - \alpha_1^{(1)}\Psi_1^*\Phi_2 - i\alpha_1^{(2)*}\hat{\Upsilon}_2\hat{\Phi}_1^* - i\alpha_1^{(2)}\hat{\Upsilon}_2^*\hat{\Phi}_1\right],\\
&\delta_1 A_i = 0\,.
\end{aligned}$$

The second supersymmetry transformation is

$$\begin{aligned}
&\delta_2\hat{\Phi}_1 = -\frac{i}{\sqrt{2mc}}\,\alpha_1^{(1)*}\hat{D}_-\hat{\Upsilon}_2\,, \qquad \delta_2\Phi_2 = -\frac{1}{\sqrt{2mc}}\,\alpha_1^{(2)*}D_+\Psi_1\,,\\
&\delta_2\Psi_1 = -\frac{1}{\sqrt{2mc}}\,\alpha_1^{(2)}D_-\Phi_2\,, \qquad \delta_2\hat{\Upsilon}_2 = \frac{i}{\sqrt{2mc}}\,\alpha_1^{(1)}\hat{D}_+\hat{\Phi}_1\,,\\
&\delta_2 A_0 = -\frac{e}{(2mc)^{3/2}\kappa}\left[\alpha_1^{(2)*}(D_+\Psi_1)\Phi_2^* - \alpha_1^{(2)}(D_+\Psi_1)^*\Phi_2\right.\\
&\qquad\qquad \left. +i\alpha_1^{(1)*}\hat{D}_-\hat{\Upsilon}_2\hat{\Phi}_1^* + i\alpha_1^{(1)}(\hat{D}_-\hat{\Upsilon}_2)^*\hat{\Phi}_1\right],\\
&\delta_2 A_+ = -\frac{2e}{\sqrt{2mc}\kappa}\left[\alpha_1^{(1)*}\hat{\Upsilon}_2\hat{\Phi}_1^* - i\alpha_1^{(2)}\Psi_1^*\Phi_2\right],\\
&\delta_2 A_- = \frac{2e}{\sqrt{2mc}\kappa}\left[\alpha_1^{(1)}\hat{\Upsilon}_2^*\hat{\Phi}_1 + i\alpha_1^{(2)*}\Psi_1\Phi_2^*\right].
\end{aligned}$$

Now we should comment on the parameters of supersymmetry transformation. Fist of all, $\alpha_2^{(1)}$ (or equivalently $\alpha_2^{(2)}$) is not contained in the NR supersymmetry. Secondly, $\alpha_1^{(1)}$ and $\alpha_1^{(2)}$ are not separated. That is, those are common to both the leading and the next-to-leading supersymmetries. According to the knowledge obtained in the previous section, we may argue that all of the next-to-leading supersymmetries are broken due to the interaction potential and all of the leading supersymmetries are preserved. Indeed, that is the case. We can check this statement explicitly by acting the supersymmetry transformations to the Lagrangian with (4.1). In addition, since there is no next-to-leading supersymmetry, we have no superconformal symmetry. In summary, we have two complex supercharges

$$Q_1^{(1)} = \sqrt{2m}\int d^2x\,\Phi_2\Psi_1^*\,, \qquad Q_1^{(2)} = \sqrt{2m}\int d^2x\,\hat{\Phi}_1\hat{\Upsilon}_2^*\,, \tag{4.8}$$

whose non-trivial Poisson brackets are given by

$$\begin{aligned}\{Q_1^{(1)}, Q_1^{(1)*}\} &= -2mi(N_{\rm B2} + N_{\rm F1})\,, \\ \{Q_1^{(2)}, Q_1^{(2)*}\} &= -2mi(N_{\rm B1} + N_{\rm F2})\,. \end{aligned} \tag{4.9}$$

Thus, we have found a less supersymmetric Schrödinger algebra in this case. That is, the number of preserved supersymmetries is different according to the choice of NR limits while the Schrödinger symmetry is still preserved.

4.2 A mixed case 2. - the PAPA case

As another possibility, let us consider the following mixed case:

$$\hat{\Phi}_1 = \Phi_2 = \hat{\Psi} = \Upsilon = 0\,,$$

which leads us to the following NR Lagrangian for the matter fields

$$\begin{aligned}\mathcal{L}_{\rm NR} = &\, i\Phi_1^* D_t\Phi_1 - \frac{1}{2m}(D_i\Phi_1)^* D_i\Phi_1 + i\hat{\Phi}_2^*\hat{D}_t\hat{\Phi}_2 - \frac{1}{2m}(\hat{D}_i\hat{\Phi}_2)^*\hat{D}_i\hat{\Phi}_2 \\ &+ i\Psi_1^* D_t\Psi_1 - \frac{1}{2m}(D_i\Psi_1)^* D_i\Psi_1 + i\hat{\Upsilon}_2^*\hat{D}_t\hat{\Upsilon}_2 - \frac{1}{2m}(\hat{D}_i\hat{\Upsilon}_2)^*\hat{D}_i\hat{\Upsilon}_2 \\ &- \frac{e}{2mc}F_{12}(|\Psi_1|^2 + |\hat{\Upsilon}_2|^2) - \lambda\left(|\Phi_1|^4 - |\hat{\Phi}_2|^4\right) \\ &+ 3\lambda\left(|\Phi_1|^2 + |\hat{\Phi}_2|^2\right)\left(|\Psi_1|^2 - |\hat{\Upsilon}_2|^2\right) \\ &- 4\lambda\left(|\Phi_1|^2|\Psi_1|^2 - |\hat{\Phi}_2|^2|\hat{\Upsilon}_2|^2\right) \\ &+ 4\lambda(\Phi_1\hat{\Phi}_2\Psi_1^*\hat{\Upsilon}_2^* + \Phi_1^*\hat{\Phi}_2^*\hat{\Upsilon}_2\Psi_1) + \mathcal{O}(1/c^2)\,. \end{aligned} \tag{4.10}$$

In the derivation of (4.10) we used the fermion equations of motion

$$\Psi_2 = -\frac{1}{2mc}D_+\Psi_1\,, \qquad \hat{\Upsilon}_1 = -\frac{1}{2mc}\hat{D}_-\hat{\Upsilon}_2 \tag{4.11}$$

and removed Ψ_2 and $\hat{\Upsilon}_1$.

Schrödinger symmetry. The Lagrangian (4.10) still has the Schrödinger symmetry. The algebra can easily be derived in the same way as in the previous section. For the bosonic generators the different point is just a sign of the charge e for anti-particles. To avoid the repetition, we will not present the computation of the algebra here.

U(1) symmetries. There are three U(1) symmetries. The corresponding generators are

$$N_{\rm B} = \int d^2x\,\left(|\Phi_1|^2 - |\hat{\Phi}_2|^2\right)\,, \qquad N_{\rm F} = \int d^2x\,\left(|\Psi_1|^2 - |\hat{\Upsilon}_2|^2\right)\,,$$

and the mass operator M.

The positivity of the hamiltonian. The original Hamiltonian is given by

$$
\begin{aligned}
H = \int d^2x \Big[& \frac{1}{2m}(D_i\Phi_1)^* D_i\Phi_1 + \frac{1}{2m}(\hat{D}_i\hat{\Phi}_2)^*\hat{D}_i\hat{\Phi}_2 \\
& +\frac{1}{2m}(D_i\Psi_1)^* D_i\Psi_1 + \frac{1}{2m}(\hat{D}_i\hat{\Upsilon}_2)^*\hat{D}_i\hat{\Upsilon}_2 \\
& +\frac{e}{2mc}F_{12}(|\Psi_1|^2 + |\hat{\Upsilon}_2|^2) + \lambda(|\Phi_1|^4 - |\hat{\Phi}_2|^4) \\
& -3\lambda(|\Phi_1|^2 + |\hat{\Phi}_2|^2)(|\Psi_1|^2 - |\hat{\Upsilon}_2|^2) \\
& +4\lambda(|\Phi_1|^2|\Psi_1|^2 - |\hat{\Phi}_2|^2|\hat{\Upsilon}_2|^2) - 4\lambda(\Phi_1\hat{\Phi}_2\Psi_1^*\hat{\Upsilon}_2^* + \Phi_1^*\hat{\Phi}_2^*\hat{\Upsilon}_2\Psi_1)\Big] .
\end{aligned}
\tag{4.12}
$$

By using the Gauss law constraint

$$
F_{12} = \frac{e}{\kappa}(|\Phi_1|^2 - |\hat{\Phi}_2|^2 + |\Psi|^2 - |\hat{\Upsilon}_2|^2)\,, \tag{4.13}
$$

this expression can be rewritten as

$$
\begin{aligned}
H = \int d^2x \Big[& \frac{1}{2m}(D_+\Phi_1)^* D_+\Phi_1 + \frac{1}{2m}(\hat{D}_-\hat{\Phi}_2)^*\hat{D}_-\hat{\Phi}_2 \\
& +\frac{1}{2m}(D_+\Psi_1)^* D_+\Psi_1 + \frac{1}{2m}(\hat{D}_-\hat{\Upsilon}_2)^*\hat{D}_-\hat{\Upsilon}_2 \\
& +4\lambda(\Phi_1\hat{\Upsilon}_2^* + \hat{\Phi}_2^*\Psi_1)(\Phi_1^*\hat{\Upsilon}_2 + \hat{\Phi}_2\Psi_1^*)\Big] .
\end{aligned}
\tag{4.14}
$$

Thus, the Hamiltonian is semi-positive definite. The conditions for the lowest energy solution are

$$
D_+\Phi_1 = \hat{D}_-\hat{\Phi}_2 = D_+\Psi_1 = \hat{D}_-\hat{\Upsilon}_2 = 0\,, \tag{4.15}
$$

and the additional constraint

$$
\Phi_1\hat{\Upsilon}_2^* = -\hat{\Phi}_2^*\Psi_1\,. \tag{4.16}
$$

Supersymmetries. The supersymmetry transformation at the leading order is given by

$$
\begin{aligned}
&\delta_1\Phi_1 = \sqrt{2mc}\,\alpha_2^{(1)*}\Psi_1\,, \qquad && \delta_1\hat{\Phi}_2 = -\sqrt{2mc}\,\alpha_2^{(1)}\hat{\Upsilon}_2\,, \\
&\delta_1\Psi_1 = -\sqrt{2mc}\,\alpha_2^{(1)}\Phi_1\,, && \delta_1\hat{\Upsilon}_2 = \sqrt{2mc}\,\alpha_2^{(1)*}\hat{\Phi}_2\,, \\
&\delta_1 A_0 = -\frac{e}{\sqrt{2mc}\,\kappa}\Big[\alpha_2^{(1)*}\Psi_1\Phi_1^* + i\alpha_2^{(2)*}\Psi_1^*\Phi_1 + \alpha_2^{(1)*}\hat{\Upsilon}_2^*\hat{\Phi}_2 + i\alpha_2^{(2)*}\hat{\Upsilon}_2\hat{\Phi}_2^*\Big]\,, \\
&\delta_1 A_i = 0\,.
\end{aligned}
$$

and the one at the next-to-leading order is

$$
\begin{aligned}
&\delta_2\Phi_1 = \frac{1}{\sqrt{2mc}}\,\alpha_2^{(2)*}D_+\Psi_1\,, \qquad && \delta_2\hat{\Phi}_2 = \frac{1}{\sqrt{2mc}}\,\alpha_2^{(2)}\hat{D}_-\hat{\Upsilon}_2\,, \\
&\delta_2\Psi_1 = \frac{1}{\sqrt{2mc}}\,\alpha_2^{(2)}D_-\Phi_1\,, && \delta_2\hat{\Upsilon}_2 = \frac{1}{\sqrt{2mc}}\,\alpha_2^{(2)*}\hat{D}_+\hat{\Phi}_2\,,
\end{aligned}
$$

$$\delta_2 A_0 = \frac{e}{(2mc)^{3/2}\kappa}\Big[\alpha_2^{(2)*}D_+\Psi_1\Phi_1^* + i\alpha_2^{(1)*}(D_+\Psi_1)^*\Phi_1$$
$$\qquad -\alpha_2^{(2)*}(\hat{D}_-\hat{\Upsilon}_2)^*\hat{\Phi}_2 - i\alpha_2^{(1)*}\hat{D}_-\hat{\Upsilon}_2\hat{\Phi}_2^*\Big]\,,$$
$$\delta_2 A_+ = -\frac{2e}{\sqrt{2mc}\,\kappa}(\alpha_2^{(1)*}\Psi_1^*\Phi_1 + \alpha_2^{(1)*}\hat{\Upsilon}_2\hat{\Phi}_2^*)\,,$$
$$\delta_2 A_- = -\frac{2ie}{\sqrt{2mc}\,\kappa}(\alpha_2^{(2)*}\Psi_1\Phi_1^* + \alpha_2^{(2)*}\hat{\Upsilon}_2^*\hat{\Phi}_2)\,. \tag{4.17}$$

First of all, note that the supersymmetry parameters $\alpha_1^{(a)}$ $(a = 1, 2)$ are decoupled from the NR supersymmetry. Then, $\alpha_2^{(1)}$ is not independent of $\alpha_2^{(2)}$ and so the number of the independent supersymmetry parameters is 2 (in real). The parameter is shared in both the leading and the next-to-leading supersymmetries. Hence we guess that only the leading supersymmetry is preserved and the next-to-leading is broken. As we can directly show, it is really the case. Since the next-to-leading supersymmetry is broken, the corresponding superconformal symmetry is also absent. In summary, the theory we have considered here has an $\mathcal{N}$=1 supersymmetry in 1+2 dimensions. The non-trivial supersymmetry algebra is given by

$$\{Q_1, Q_1^*\} = -2iM\,, \tag{4.18}$$

where the supercharge is given by

$$Q_1 = \sqrt{2m}\int d^2x\,\left[\Phi_1\Psi_1^* - \hat{\Phi}_2^*\hat{\Upsilon}_2\right]\,. \tag{4.19}$$

4.3 Peculiar features of NR supersymmetry

Summarizing the results obtained in this section, we may deduce the following statements:

1) A Schrödinger symmetry always appears independently of the choice of NR limits, but the number of preserved supersymmetries depends on the choice.

2) The NR limit with only the particles (or anti-particles) leads us to the same number of the supersymmetries as the original relativistic theory. When including anti-particles, some of the supersymmetries are broken or the supersymmetries are completely broken.

3) The supersymmetries in the original relativistic theory are not enhanced after the NR limit.

4) If supersymmetry parameters are not separated after taking a NR limit, the leading supersymmetries are preserved but the next-to-leading ones are broken.

It would be interesting to check the observations for other models.

5. NR limit as consistent truncation

Let us describe a consistency of NR limits in detail from the viewpoint of the parent relativistic theory i.e. the $\mathcal{N}$=3 CSM system. The discussion here confirms the consistency of the non-relativistic limit taken in the previous sections.

In [16], the authors addressed a consistency of NR limits in the bosonic CSM system (Jackiw-Pi model). The original relativistic action was expanded by using the field expansion like in (3.1), and the conservation of the particle number and that of the anti-particle number were checked. Since the anti-particle number is conserved independently of the particle number, we can pick up a subsector with no anti-particle.

The similar argument should be applied for supersymmetric CSM systems. The U(1) symmetries realized in the expanded action play an important role in this argument. In the case of the $\mathcal{N}$=2 CSM system [40] there are four U(1) symmetries and the number of each of particles and anti-particles is conserved independently. Thus, there is no problem to take any desired non-relativistic limit. However, the $\mathcal{N}$=3 CSM system has a more complicated potential and we should be careful about the U(1) symmetries.

U(1) symmetries and conserved quantities. In order to check whether the conditions are satisfied or not, we have to derive the expanded Lagrangian by substituting (3.1) into (2.1) without dropping off any of the particles and anti-particles.

The expanded Lagrangian at the first order of $1/c$ is composed of the kinetic terms for the bosons and fermions, $\mathcal{L}_{\rm B}$ and $\mathcal{L}_{\rm F}$, respectively, and the Pauli interaction $\mathcal{L}_{\rm Pauli}$, the four-boson interaction $\mathcal{L}_{\rm 4B}$ and the boson-fermion interaction $\mathcal{L}_{\rm BF}$ as follows:

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\rm B} + \mathcal{L}_{\rm F} + \mathcal{L}_{\rm Pauli} + \mathcal{L}_{\rm 4B} + \mathcal{L}_{\rm BF} \,, \\
\mathcal{L}_{\rm B} &= i\Phi_1^* D_t \Phi_1 + i\hat{\Phi}_1^* \hat{D}_t \hat{\Phi}_1 - \frac{1}{2m}\Big((D_i\Phi_1)^* D_i\Phi_1 + (\hat{D}_i\hat{\Phi}_1)^* \hat{D}_i\hat{\Phi}_1\Big) \\
&\quad + i\Phi_2^* D_t \Phi_2 + i\hat{\Phi}_2^* \hat{D}_t \hat{\Phi}_2 - \frac{1}{2m}\Big((D_i\Phi_2)^* D_i\Phi_2 + (\hat{D}_i\hat{\Phi}_2)^* \hat{D}_i\hat{\Phi}_2\Big) \,, \\
\mathcal{L}_{\rm F} &= i\Psi_1^* D_t \Psi_1 + i\hat{\Psi}_1^* \hat{D}_t \hat{\Psi}_1 - \frac{1}{2m}\Big((D_i\Psi)_1^* D_i\Psi_1 + (\hat{D}_i\hat{\Psi}_1)^* \hat{D}_i\hat{\Psi}_1\Big) \\
&\quad + i\Upsilon_2^* D_t \Upsilon_2 + i\hat{\Upsilon}_2^* \hat{D}_t \hat{\Upsilon}_2 - \frac{1}{2m}\Big((D_i\Upsilon_2)^* D_i\Upsilon_2 + (\hat{D}_i\hat{\Upsilon}_2)^* \hat{D}_i\hat{\Upsilon}_2\Big) \,, \\
\mathcal{L}_{\rm Pauli} &= -\frac{e}{2mc} F_{12}\Big(|\Psi_1|^2 - |\Upsilon_2|^2 - |\hat{\Psi}_1|^2 + |\hat{\Upsilon}_2|^2\Big) \,, \\
\mathcal{L}_{\rm 4B} &= -\lambda\Bigg[\Big(|\Phi_a|^2 + |\hat{\Phi}_a|^2\Big)\Big(|\Phi_1|^2 - |\Phi_2|^2 + |\hat{\Phi}_1|^2 - |\hat{\Phi}_2|^2\Big) \\
&\qquad\qquad + 2(|\Phi_1|^2|\hat{\Phi}_1|^2 - |\Phi_2|^2|\hat{\Phi}_2|^2)\Bigg] \,,
\end{aligned}
\tag{5.1}
$$

	Φ_1	$\hat\Phi_1$	Φ_2	$\hat\Phi_2$	Ψ_1	$\hat\Psi_1$	Υ_2	$\hat\Upsilon_2$
$\mathrm{U}(1)_1$	1	0	0	0	1	0	0	0
$\mathrm{U}(1)_2$	0	0	1	0	0	0	1	0
$\mathrm{U}(1)_3$	0	0	0	1	0	0	0	1
$\mathrm{U}(1)_4$	0	1	0	0	0	1	0	0

Table 2: The remaining four U(1) symmetries.

$$\begin{aligned}
\mathcal{L}_{\rm BF} =\ & 3\lambda\left(|\Phi_a|^2+|\hat\Phi_a|^2\right)\left(|\Psi_1|^2-|\Upsilon_2|^2+|\hat\Psi_1|^2-|\hat\Upsilon_2|^2\right)\\
&+4\lambda\Big[-|\Phi_1|^2|\Psi_1|^2+\Phi_1\hat\Phi_2\Psi_1^*\hat\Upsilon_2^*-\Phi_1^*\hat\Phi_2^*\Psi_1\hat\Upsilon_2+|\hat\Phi_2|^2|\hat\Upsilon_2|^2\\
&\qquad-|\hat\Phi_1|^2|\hat\Psi_1|^2+\hat\Phi_1^*\Phi_2^*\hat\Psi_1\Upsilon_2+\Phi_2\hat\Phi_1\Upsilon_2^*\hat\Psi_1^*+|\Phi_2|^2|\Upsilon_2|^2\\
&\qquad\quad-|\Phi_1|^2|\hat\Psi_1|^2+|\Phi_2|^2|\hat\Upsilon_2|^2-|\hat\Phi_1|^2|\Psi_1|^2+|\hat\Phi_2|^2|\Upsilon_2|^2\Big]\\
&-2i\lambda\Big[2\Phi_1\hat\Phi_1^*\Psi_1^*\hat\Psi_1+\Phi_1\Phi_2\Psi_1^*\Upsilon_2^*+\hat\Phi_1^*\hat\Phi_2^*\hat\Psi_1\hat\Upsilon_2+2\Phi_2\hat\Phi_2^*\Upsilon_2^*\hat\Upsilon_2\\
&\qquad+2\Phi_1^*\hat\Phi_1\Psi_1\hat\Psi_1^*+\Phi_1^*\Phi_2^*\Psi_1\Upsilon_2+\hat\Phi_1\hat\Phi_2\hat\Psi_1^*\hat\Upsilon_2^*+2\Phi_2^*\hat\Phi_2\Upsilon_2\hat\Upsilon_2^*\Big]\,. \qquad (5.2)
\end{aligned}$$

Here we have used the fermion equations of motion and removed Ψ_2 , $\hat\Psi_2$, Υ_1 and $\hat\Upsilon_1$.

From the Lagrangian (5.1) we can figure out the U(1) symmetries. The sensitive interactions are $+4\lambda[\ldots]$ and $-2i\lambda[\ldots]$ in (5.2). Without these interactions there are eight U(1) symmetries and the number of each field is conserved. In particular, the $\mathcal{N}$=2 CSM system has no problem. The interaction $+4\lambda[\ldots]$ yields eight U(1) symmetries in a non-trivial way, but $-2i\lambda[\ldots]$ breaks half of the U(1)'s. As a consequence, the four U(1) symmetries are preserved (table 2).

From $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$ the total number of Φ_1 , Φ_2 , Ψ_1 and Υ_2 is conserved. The total number of $\hat\Phi_1$, $\hat\Phi_2$, $\hat\Psi_1$ and $\hat\Upsilon_2$ is also conserved from $\mathrm{U}(1)_3$ and $\mathrm{U}(1)_4$. Hence, by setting the latter number to be zero, we can realize all particle case (the PPPP case) consistently as in [16]. By using $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_3$, we understand that the PAPA case is also consistent to the argument in [16].

However, only from the viewpoint of the consistent truncation of the matter content, one may allow a weaker condition. We propose the following two criteria for the consistent truncation of the fields in NR limits:

1. The total number of the matter fields picked up in the NR limit is conserved.
   (strong condition)

2. The truncation of the field is consistent to all of the equations of motion.
   (weak condition)

Note that if the strong condition is satisfied then the weak condition is also satisfied.

The PPPP case and the PAPA case satisfy the strong condition but the APPA case does not. Still, the APPA case satisfies the weak condition. Hence the truncation is consistent at the level of the equations of motion and it would still make sense at least at the classical level. There would possibly be a subtlety at quantum mechanical level.

On the other hand, an exotic case such as the PPPA case is excluded even by the weak condition. In addition, the supersymmetries are completely broken.

Here, we have discussed the consistency of the matter field truncation in NR limits. Nevertheless, it might be possible to consider a wider class of NR limits if we take the stance to regard the NR limit as a generation technique of super Schrödinger invariant field theories. It would be interesting to investigate whether the consistency of NR limits is related to the consistency of the resulting NR theories.

6. Summary and discussion

We have presented new super Schrödinger invariant field theories by considering NR limits of the $\mathcal{N}$=3 relativistic CSM system in 1+2 dimensions.

First, by taking a NR limit with only the particles, we have derived an $\mathcal{N}$=3 super Schrödinger invariant CSM system. By using the standard Noether theorem, we explicitly constructed the generators of the super Schrödinger symmetry and computed the Poisson brackets of the generators.

Then, as other NR limits we have considered the two mixed cases: 1) the APPA case and 2) the PAPA case. The bosonic Schrödinger symmetry still persists in both cases but the number of the preserved supersymmetries decreased from $\mathcal{N}$=3 to $\mathcal{N}$=2 and $\mathcal{N} = 1$ respectively. In particular, it is possible to find the supersymmetries only at the leading, that is, no supersymmetries at the next-to-leading.

An interesting observation is that a less supersymmetric Schrödinger algebra is realized depending on the matter content held on in taking the NR limit. In any NR limits the bosonic Schrödinger symmetry is always preserved. On the other hand, the number of the preserved supersymmetries depends on the matter content in the NR limit. In particular, the inclusion of anti-particles would be sensitive only to supersymmetries.

It would be interesting to study the other NR limits we have not discussed here. In principle, it can be done and several super Schrödinger algebras would be obtained. As another direction, it would be interesting to investigate the NR symmetry in a broken phase where $\langle\phi\rangle \neq 0$, though we have discussed only in a symmetric phase where $\langle\phi\rangle = 0$.

The full quantum treatment of the CSM theory should be investigated further. In particular, the full Schrödinger invariance might be broken by the non-zero beta function for various coupling constants. With enough supersymmetries ($\mathcal{N} = 2$ or higher), we can argue that the beta function for our models vanish to all orders in perturbation theory. Thanks to the supersymmetry, all the coupling constants are related to the charge e^2/κ of the Chern-Simons theory, so we only have to study the renormalization of the photon polarization function. The perturbative corrections to the photon polarization function, however, trivially vanish in the non-relativistic system just because it does not allow particle/anti-particle pair creation, which can easily be seen from the retarded nature of the Green functions for matters. This guarantees the vanishing beta function for the supersymmetric theories to all orders in perturbation theory. One can also explicitly check that the perturbative beta function vanishes [47, 48].[6] It would be interesting to investigate further

[6]One of the author Y.N. would like to thank Y. Nishida for discussions on this point.

the situations in less supersymmetric theories descended from the same supersymmetric parent theory.

The $\mathcal{N}$=3 NR CSM systems and their relatives, which contain a single gauge field, have been studied well here. The next is to study the CSM system containing two gauge fields like [43 – 45]. It would be nice to investigate NR limits of the ABJM theory [45]. We will report on this issue in the near future [49].

We hope that our method would be a clue to develop super Schrödinger invariant field theories and could formulate some basic techniques there.

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A. Dimensional analysis

In order to take a NR limit, we need to recover the speed of light c, so let us check dimensions of the fields and parameters contained in a relativistic CSM system. For this purpose, it is enough to consider the $\mathcal{N}$=2 CSM system [41] since the terms appearing in the $\mathcal{N}$=3 CSM system are almost the same as in the $\mathcal{N}$=2 CSM system.

First of all, let us set $[\hbar] = 1$. Since

$$[\hbar] = ML^2T^{-1} = 1\,,$$

we obtain

$$[M] = TL^{-2}\,.$$

Accordingly, the action is dimensionless.

From the kinematic term $\int dtd^2x\partial\phi\partial\phi$, the dimension of the scalar field ϕ is

$$[\phi] = L^{-1/2}\,.$$

Note that the mass term $\int dtd^2xm^2c^2\phi^2$ is consistent.

Next let us consider the self-interaction terms of ϕ. From the quartic interaction $\int dtd^2x\frac{me^2}{c\kappa}|\phi|^4$, we can figure out

$$\left[\frac{e^2}{\kappa}\right] = L^2T^{-2}\,.$$

Thus, the 6th order term $\int dtd^2x\frac{e^4}{c^4\kappa^2}|\phi|^6$ is consistent.

Now let us consider the dimension of the charge e from the Coulomb force F in 1+2 dimensions:

$$F = ma = -\frac{e^2}{r}\,.$$

It is easy to see that

$$[e^2] = T^{-1} \qquad \text{and} \qquad [\kappa] = TL^{-2}\,.$$

Note that the CS coupling is dimensionful.

Then, from the covariant derivative $D_\mu = \partial_\mu + i\frac{e}{c}A_\mu$, we find

$$[A_\mu] = T^{-1/2}\,.$$

Thus, the CS terms are also consistent.[7] For the fermionic field, the argument is the similar.

Non-relativistic case. After taking a NR limit, it is natural to set $[M]=1$ and then

$$[T] = [L^2]\,.$$

This is appropriate to a NR field theory with dynamical exponent $z = 2$ such as a Schrödinger invariant field theory . Note that the CS coupling κ becomes dimensionless.

B. The detail of the spinor rotation

In the original paper [42] the gamma matrices are given in the Majorana representation

$$\gamma^0_{\rm M} = -i\sigma_2\,, \quad \gamma^1_{\rm M} = \sigma_3\,, \quad \gamma^2_{\rm M} = \sigma_1\,,$$

while each of the two fermion fields is a 2-component complex fermion defined as

$$\begin{pmatrix} \psi \\ \chi \end{pmatrix} = \psi_1 + i\psi_2\,,$$

where ψ_k $(k=1,2)$ are real 4-component Majorana fermions i.e., $(\psi_k)^* = \psi_k$.

In order to take a non-relativistic limit, it is convenient to move from the Majorana representation to the Dirac one (2.3) . This can be done by using the following transformation:

$$\gamma^\mu = U^{-1}\gamma^\mu_{\rm M}U\,, \qquad U = U_1U_2\,, \quad U_1 = \frac{1}{\sqrt{2}}(1+i\sigma_1)\,, \quad U_2 = \frac{1}{\sqrt{2}}(1+i\sigma_3)\,.$$

The following relations are available:

$$U^TU = i\sigma_1\,, \qquad U^{-1}U^* = -i\sigma_1\,.$$

Associated with this rotation, the original complex fermions $\psi_{\rm o}$ and $\chi_{\rm o}$ are also rotated to a new set of the complex fermions ψ and χ in the Lagrangian.

[7] In (2.1a) of [40], $\kappa/4c$ should be replaced by $\kappa/4$. This replacement is consistent to (2.8) of [40] .

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The condition for α_2 — The rotated Majorana condition. We should be careful for the rotation in rewriting the supersymmetry transformation because a part of the supersymmetry parameters is given in terms of Majorana spinor in the original paper [42].

To see the connection between their Majorana condition and our condition, we note that our spinor α_2 is related to α_{o2} as

$$\alpha_2 = \begin{pmatrix} \alpha_2^{(1)} \\ \alpha_2^{(2)} \end{pmatrix} = U^{-1}\alpha_{o2} = U^{-1}\begin{pmatrix} \alpha_{o2}^{(1)} \\ \alpha_{o2}^{(2)} \end{pmatrix} .$$

By defining new real spinors

$$\alpha_o^{(\pm)} \equiv \frac{1}{2}\left(\alpha_{o2}^{(1)} \pm \alpha_{o2}^{(2)}\right) ,$$

we obtain

$$\alpha_2^{(1)} = \alpha_o^{(-)} - i\alpha_o^{(+)} , \qquad \alpha_2^{(2)} = \alpha_o^{(+)} - i\alpha_o^{(-)} .$$

Thus, in our notation, the Majorana condition reads

$$\alpha_2^{(2)} = -i(\alpha_2^{(1)})^* , \tag{B.1}$$

and $\alpha_2^{(1)}$ and $\alpha_2^{(2)}$ are not independent. As an immediate consequence, the number of independent components of α_2 is 2 in real.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] S.S. Gubser, *Breaking an abelian gauge symmetry near a black hole horizon*, arXiv:0801.2977.

[5] S.A. Hartnoll, C.P. Herzog and G.T. Horowitz, *Building a holographic superconductor*, *Phys. Rev. Lett.* **101** (2008) 031601 [arXiv:0803.3295]; *Holographic superconductors*, *JHEP* **12** (2008) 015 [arXiv:0810.1563].

[6] E. Keski-Vakkuri and P. Kraus, *Quantum Hall effect in AdS/CFT*, *JHEP* **09** (2008) 130 [arXiv:0805.4643].

[7] J.L. Davis, P. Kraus and A. Shah, *Gravity dual of a quantum Hall plateau transition*, *JHEP* **11** (2008) 020 [arXiv:0809.1876].

[8] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602 [hep-th/0603001]; *Aspects of holographic entanglement entropy*, *JHEP* **08** (2006) 045 [hep-th/0605073].

[9] S.S. Gubser and S.S. Pufu, *The gravity dual of a p-wave superconductor*, *JHEP* **11** (2008) 033 [arXiv:0805.2960].

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[10] M.M. Roberts and S.A. Hartnoll, *Pseudogap and time reversal breaking in a holographic superconductor*, *JHEP* **08** (2008) 035 [arXiv:0805.3898].

[11] K. Maeda and T. Okamura, *Characteristic length of an AdS/CFT superconductor*, *Phys. Rev.* **D 78** (2008) 106006 [arXiv:0809.3079].

[12] M. Fujita, Y. Hikida, S. Ryu and T. Takayanagi, *Disordered systems and the replica method in AdS/CFT*, *JHEP* **12** (2008) 065 [arXiv:0810.5394].

[13] S. Sachdev and M. Mueller, *Quantum criticality and black holes*, arXiv:0810.3005.

[14] C.R. Hagen, *Scale and conformal transformations in galilean-covariant field theory*, *Phys. Rev.* **D 5** (1972) 377.

[15] U. Niederer, *The maximal kinematical invariance group of the free Schrödinger equation*, *Helv. Phys. Acta* **45** (1972) 802.

[16] R. Jackiw and S.Y. Pi, *Soliton solutions to the gauged nonlinear Schrödinger equation on the plane*, *Phys. Rev. Lett.* **64** (1990) 2969; *Classical and quantal nonrelativistic Chern-Simons theory*, *Phys. Rev.* **D 42** (1990) 3500 [*Erratum ibid.* **48** (1993) 3929].

[17] M. Henkel, *Schrödinger invariance in strongly anisotropic critical systems*, *J. Stat. Phys.* **75** (1994) 1023 [hep-th/9310081].

[18] T. Mehen, I.W. Stewart and M.B. Wise, *Conformal invariance for non-relativistic field theory*, *Phys. Lett.* **B 474** (2000) 145 [hep-th/9910025].

[19] D.T. Son and M. Wingate, *General coordinate invariance and conformal invariance in nonrelativistic physics: unitary Fermi gas*, *Ann. Phys. (NY)* **321** (2006) 197 [cond-mat/0509786].

[20] Y. Nishida and D.T. Son, *Nonrelativistic conformal field theories*, *Phys. Rev.* **D 76** (2007) 086004 [arXiv:0706.3746].

[21] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schrödinger symmetry*, *Phys. Rev.* **D 78** (2008) 046003 [arXiv:0804.3972].

[22] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, *Phys. Rev. Lett.* **101** (2008) 061601 [arXiv:0804.4053].

[23] W.D. Goldberger, *AdS/CFT duality for non-relativistic field theory*, arXiv:0806.2867.

[24] J.L.B. Barbon and C.A. Fuertes, *On the spectrum of nonrelativistic AdS/CFT*, *JHEP* **09** (2008) 030 [arXiv:0806.3244].

[25] W.-Y. Wen, *AdS/NRCFT for the (super) Calogero model*, arXiv:0807.0633.

[26] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up galilean holography*, *JHEP* **11** (2008) 080 [arXiv:0807.1099].

[27] J. Maldacena, D. Martelli and Y. Tachikawa, *Comments on string theory backgrounds with non-relativistic conformal symmetry*, *JHEP* **10** (2008) 072 [arXiv:0807.1100].

[28] A. Adams, K. Balasubramanian and J. McGreevy, *Hot spacetimes for cold atoms*, *JHEP* **11** (2008) 059 [arXiv:0807.1111].

[29] S. Kachru, X. Liu and M. Mulligan, *Gravity duals of Lifshitz-like fixed points*, *Phys. Rev.* **D 78** (2008) 106005 [arXiv:0808.1725].

JHEP01(2009)006

[30] S.A. Hartnoll and K. Yoshida, *Families of IIB duals for nonrelativistic CFTs*, *JHEP* **12** (2008) 071 [arXiv:0810.0298].

[31] M. Schvellinger, *Kerr-AdS black holes and non-relativistic conformal QM theories in diverse dimensions*, *JHEP* **12** (2008) 004 [arXiv:0810.3011].

[32] L. Mazzucato, Y. Oz and S. Theisen, *Non-relativistic branes*, arXiv:0810.3673.

[33] M. Rangamani, S.F. Ross, D.T. Son and E.G. Thompson, *Conformal non-relativistic hydrodynamics from gravity*, arXiv:0811.2049.

[34] J.P. Gauntlett, J. Gomis and P.K. Townsend, *Supersymmetry and the physical phase space formulation of spinning particles*, *Phys. Lett.* **B 248** (1990) 288.

[35] C. Duval and P.A. Horvathy, *On Schrödinger superalgebras*, *J. Math. Phys.* **35** (1994) 2516 [hep-th/0508079].

[36] M. Henkel and J. Unterberger, *Supersymmetric extensions of Schrödinger-invariance*, *Nucl. Phys.* **B 746** (2006) 155 [math-ph/0512024].

[37] M. Sakaguchi and K. Yoshida, *Super Schrödinger algebra in AdS/CFT*, *J. Math. Phys.* **49** (2008) 102302 [arXiv:0805.2661].

[38] M. Sakaguchi and K. Yoshida, *More super Schrödinger algebras from* $PSU(2,2|4)$, *JHEP* **08** (2008) 049 [arXiv:0806.3612].

[39] Y. Nakayama, *Index for non-relativistic superconformal field theories*, *JHEP* **10** (2008) 083 [arXiv:0807.3344].

[40] M. Leblanc, G. Lozano and H. Min, *Extended superconformal Galilean symmetry in Chern-Simons matter systems*, *Ann. Phys. (NY)* **219** (1992) 328 [hep-th/9206039].

[41] C.-k. Lee, K.-M. Lee and E.J. Weinberg, *Supersymmetry and selfdual Chern-Simons systems*, *Phys. Lett.* **B 243** (1990) 105.

[42] H.-C. Kao and K.-M. Lee, *Selfdual Chern-Simons systems with an* $N=3$ *extended supersymmetry*, *Phys. Rev.* **D 46** (1992) 4691 [hep-th/9205115].

[43] D. Gaiotto and E. Witten, *Janus configurations, Chern-Simons couplings, and the* Θ*-angle in* $N=4$ *super Yang-Mills theory*, arXiv:0804.2907.

[44] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, $N=4$ *superconformal Chern-Simons theories with hyper and twisted hyper multiplets*, *JHEP* **07** (2008) 091 [arXiv:0805.3662]; $N=5,6$ *superconformal Chern-Simons theories and* $M2$*-branes on orbifolds*, *JHEP* **09** (2008) 002 [arXiv:0806.4977].

[45] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, $N=6$ *superconformal Chern-Simons-matter theories,* $M2$*-branes and their gravity duals*, *JHEP* **10** (2008) 091 [arXiv:0806.1218].

[46] C. Duval, P.A. Horvathy and L. Palla, *Spinors in non-relativistic Chern-Simons electrodynamics*, *Ann. Phys. (NY)* **249** (1996) 265 [hep-th/9510114].

[47] O. Bergman and G. Lozano, *Aharonov-Bohm scattering, contact interactions and scale invariance*, *Ann. Phys. (NY)* **229** (1994) 416 [hep-th/9302116].

[48] S.-J. Kim and C.-k. Lee, *Quantum description of anyons: role of contact terms*, *Phys. Rev.* **D 55** (1997) 2227 [hep-th/9606054].

[49] Y. Nakayama, M. Sakaguchi and K. Yoshida, *Non-relativistic* $M2$*-brane gauge theory and new superconformal algebra*, in preparation.